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A PHRAGMÉN-LINDELÖF THEOREM CONJECTURED BY D. J. NEWMAN

BY
W. H. J. FUCHS[1]

In memory of H. C. Wang

ABSTRACT. Let D be a region of the complex plane, $\infty \in \partial D$. If $f(z)$ is holomorphic in D, write $M(r) = \sup_{|z|=r, z\in D} |f(z)|$.

THEOREM 1. *If $f(z)$ is holomorphic in D and* $\limsup_{z\to\zeta, z\in D} |f(z)| < 1$ *for $\zeta \in \partial D$, $\zeta \neq \infty$, then one of the following holds* (a) $|f(z)| < 1$ $(z \in D)$, (b) *$f(z)$ has a pole at ∞*, (c) $\log M(r)/\log r \to \infty$ *as $r \to \infty$. If $M(r)/r \to 0$ $(r \to \infty)$, then* (a) *must hold.*

1. The main result of this paper is a proof of a conjecture of D. J. Newman.

Throughout the paper D is an unbounded region (= connected open set) of the z-plane with at least one boundary point $\neq \infty$, $f(z)$ is holomorphic in D and for every boundary point $\zeta \neq \infty$

$$\limsup_{\substack{z\to\zeta\\ z\in D}} |f(z)| < 1. \tag{1.1}$$

Let $M(r) = \sup_{z\in D, |z|=r} |f(z)|$.

THEOREM 1. *Under the conditions on D and $f(z)$ stated above one of the following three mutually exclusive possibilities must occur*:

(a) $|f(z)| \leqslant 1$ $(z \in D)$.

(b) *$f(z)$ has a pole at ∞.*

(c) $\log M(r)/\log r \to \infty$ $(r \to \infty)$.

THEOREM 2 (NEWMAN'S CONJECTURE). *Under the assumptions of Theorem* 1

$$\liminf M(r)/r = 0 \qquad (r \to \infty) \tag{1.2}$$

implies

$$|f(z)| \leqslant 1 \qquad (z \in D).$$

This is an immediate consequence of Theorem 1, since (1.2) excludes the possibilities (b) and (c).

An interesting feature of Theorem 2 is that the growth condition (1.2) is entirely independent of the geometry of D. Another feature that distinguishes Theorem 2 from other theorems of Phragmén-Lindelöf type is the following: The standard

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method of proving such theorems is to translate them into a statement about the subharmonic function $\log|f(z)|$ and then to prove this statement for subharmonic functions by an application of the theory of harmonic measure. This is not possible in our case: The translation to subharmonic functions is: "If $u(z)$ is subharmonic in the region D,

$$\limsup_{z\to\zeta,\, z\in D} u(z) \leqslant 0 \qquad (\zeta \in \partial D, \zeta \neq \infty)$$

and

$$\liminf_{r\to\infty} \left\{ \sup_{|z|=r,\, z\in D} u(z) - \log r \right\} = -\infty,$$

then $u(z) \leqslant 0$ in D."

But this statement is not always true, as the case $D = \{z\colon |z| > 1\}$, $u(z) = \frac{1}{2}\log|z|$ shows.

To get a conclusion for the subharmonic case it is convenient to bring the 'exceptional boundary point' $z = \infty$ into the finite plane. Without any loss of generality we may assume that $0 \in D$. Inversion $z \to 1/z$ changes D into a region B which contains a full neighborhood of ∞. $z = \infty$ is thrown to the origin.

THEOREM 3. *Let B be a region containing $z = \infty$ and let $z = 0$ belong to the boundary ∂B of B.*

Let $u(z)$ be subharmonic in B and let $\limsup_{z\to\zeta;\, z\in B} u(z) \leqslant 0$ $(\zeta \in \partial B, \zeta \neq 0)$. *Then*

$$u(z) \leqslant C\log|z|^{-1}\left(z \in B, |z| < \tfrac{1}{2}\right) \text{ implies } u(z) \leqslant 0\,(z \in B), \tag{1.3}$$

if and only if ∂B has positive logarithmic capacity and $z = 0$ is a regular boundary point for the Dirichlet problem in B.

2. Proof of Theorem 1. We must prove that either (a) or (b) holds, if (c) is false, i.e. if

$$\liminf \log M(r)/\log r < p < \infty. \tag{2.1}$$

If we suppose that (a) is false, then

$$1 < \sup_{z\in D} |f(z)| = \mu \leqslant \infty. \tag{2.2}$$

Since $f'(z)$ has at most a denumerable set of zeros in D, there is at most a denumerable set F of A, $1 < A < \mu$, such that one of the level curves,

$$|f(z)| = A \qquad (z \in D), \tag{2.3}$$

contains a zero of f'. If $A \notin F$, each curve on which (2.3) holds is a simple curve, either closed or without a finite endpoint. We choose an $A \notin F$ and consider the nonempty open set

$$D_A = \{z\colon z \in D, |f(z)| > A\}. \tag{2.4}$$

Each component of D_A must be unbounded, by the maximum modulus principle. We consider two cases.

Case 1. There is an unbounded level curve (2.3). We shall show that in this case (a) must hold. Choose a point b on this level curve at minimum distance from the origin and let $\mathcal{C}$ be one of the two unbounded portions into which b divides the level curve. For $R > |b|$ let $\mathcal{C}_R$ be the portion of $\mathcal{C}$ between b and the first point of intersection of $\mathcal{C}$ with $|z| = R$.

Let $\omega_R(z)$ be the harmonic measure of $|z| = R$ with respect to the region $\Delta_R = \{|z| < R\} \setminus \mathcal{C}_R$.

The Carleman-Milloux theorem [**1**, p. 108] asserts that

$$\omega_R(z) \leqslant \varphi_R(-|z|) \qquad (z \leqslant R), \tag{2.5}$$

where $\varphi_R(z)$ is the harmonic measure of $\{|z| = R\}$ with respect to

$$\{|z| < R\} \setminus \{z = x + iy : y = 0, |b| \leqslant x < R\}.$$

A direct calculation of φ_R shows that

$$\varphi_R(-|z|) < K(z) R^{-1/2} \qquad (z \text{ fixed}, R \to \infty). \tag{2.6}$$

The function $u(z) = \log|f(z)| - \log M(R)\omega_R(z) - \log A$ is harmonic in $\{|z| < R\} \cap D_A \subset \Delta_R$. At every boundary point of $\{|z| < R\} \cap D_A$, $u(z) \leqslant 0$ and therefore

$$u(z) \leqslant 0, \qquad z \in \{|z| < R\} \cap D_A. \tag{2.7}$$

Keep $z \in D_A$ fixed and let $R \to \infty$ through a sequence of values for which $\log M(R) < p \log R$. This is possible, by (2.1). Then it follows from (2.7), (2.5) and (2.6) that

$$\log|f(z)| \leqslant \log A \qquad (z \in D_A).$$

But this contradicts (2.4) and shows that (2.2) is not tenable; we have the alternative (a) of Theorem 1.

Case 2. All level curves on which (2.3) holds are bounded. We shall prove that there is a positive ρ such that

$$\{\rho < |z| < \infty\} \subset D_A.$$

It will then follow easily that under the hypothesis (2.1) either (a) or (b) must be true.

We show first that only a finite number of components $\mathcal{C}_n$ of (2.3) can meet a circle $|z| = r$ which intersects D_A:

The intersection of $|z| = r$ and D is the union of disjoint, open, circular arcs I. The function

$$g(z) = f(z)\overline{f(r^2/\bar{z})}$$

is holomorphic in a neighborhood of any closed subarc J of an arc I and, on $|z| = r$, $g(z) = |f(z)|^2$.

Suppose that an infinity of distinct level curves $\mathcal{C}_n$ given by (2.3) intersect $|z| = r$. Let z_n be a point of intersection of $\mathcal{C}_n$ with $|z| = r$. Without loss of generality we may assume that $z_n \to Z$, $|Z| = r$, $g(Z) = A^2 = g(z_n)$. Either $Z \in D$ or $Z \in \partial D$. The latter possibility can be excluded, since every boundary point of D other than ∞ has a neighborhood in which $|f(z)| < \frac{1}{2}(1 + A) < A$. Therefore Z is an (interior) point of one of the intervals I described above and $g(z)$ is holomorphic in a neighborhood of Z. For sufficiently large n, z_n belongs to this neighborhood and, because of $g(z_n) = g(Z) = A^2$, $g(z) = |f(z)|^2 = A^2$ on I. But this contradicts the assumption that distinct components of (2.3) pass through $z_1, z_2, \ldots$; only a finite number of components of (2.3) can meet $|z| = r$.

Now it is easy to see that D_A is a region: Since every component of D_A is unbounded, it is enough to show that two points $z_1, z_2 \in D_A$ with $|z_1| = |z_2| = r$ can be connected by a curve lying in D_A. If one of the two arcs of $|z| = r$ with endpoints z_1, z_2 lies in D_A we have nothing to prove. If this is not the case, choose one of these two arcs. It intersects the components $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ of (2.3) all of which are smooth simple curves with a minimum distance $\delta > 0$ from each other; z_1 and z_2 can be connected by a path consisting of parts of $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_m$ and circular arcs of $|z| = r$ which lie in D_A. A slight displacement of the path into D_A will produce the desired curve joining z_1 and z_2 in D_A.

Let $D(r) = D_A \cap \{c < |z| < r\}$.

For sufficiently large c the boundary $\partial D(r)$ of $D(r)$ will consist of arcs of $|z| = c$ and $|z| = r$ which lie in D_A and of portions of level curves (2.3).

We apply Gauss's Theorem to the harmonic function $\log|f(z)|$ in $D(r)$:

$$\int_{\partial D(r)} \frac{\partial}{\partial n} \log|f(z)|\, |ds| = 0. \tag{2.8}$$

Here $\partial/\partial n$ denotes the differentiation in the direction of the outward normal to $\partial D(r)$ and ds is the element of arc length. On the level curves (2.3)

$$(\partial/\partial n)\log|f(z)| \leqslant 0. \tag{2.9}$$

Equality in (2.9) cannot hold for z on a level curve $\mathcal{C}$ satisfying (2.3), because on such a curve $(\partial/\partial s)\log|f(z)| = 0$ ($\partial/\partial s$ = differentiation in a direction tangential to $\mathcal{C}$) while

$$|f'(z)| = \left| \frac{\partial}{\partial n} \log|f(z)| - i\frac{\partial}{\partial s} \log|f(z)| \right| |f(z)| \neq 0.$$

Therefore on any such level curve $\mathcal{C}$

$$0 > \int_{\mathcal{C}} \frac{\partial}{\partial n} \log|f(z)|\, |ds|. \tag{2.10}$$

By the Cauchy-Riemann equations

$$\int_{\mathcal{C}} \frac{\partial}{\partial n} \log|f(z)|\, |ds| = \text{change of } \arg f(z) \text{ as } \mathcal{C} \text{ is described once in the clockwise direction.}$$

By the single-valuedness of $f(z)$ the expression on the right-hand side of the last equation must be an integer multiple of 2π. Hence, by (2.10), on any closed level curve,

$$\int_{\mathcal{C}} \frac{\partial}{\partial n} \log|f(z)|\, |ds| \leqslant -2\pi. \tag{2.11}$$

Let $\nu(r)$ be the number of components of (2.3) entirely situated in $c < |z| < r$. By (2.8), (2.9) and (2.11)

$$0 \leqslant -2\pi\nu(r) + \int_{re^{i\theta} \in D_A} \frac{\partial}{\partial r}\log|f(re^{i\theta})|\,d\theta$$

$$- \int_{ce^{i\theta} \in D_A} \left[\frac{\partial}{\partial r}\log|f(re^{i\theta})|\right]_{r=c} d\theta$$

$$\leqslant -2\pi\nu(r) + \int_{re^{i\theta} \in D_A} \frac{\partial}{\partial r}\log|f(re^{i\theta})|\,d\theta + K \qquad (K = K(c)).$$

Divide by r, integrate from c to ρ:

$$2\pi\int_c^\rho \frac{\nu(r)}{r}\,dr \leqslant K\log(\rho/c) + \int\int_{re^{i\theta} \in D(\rho)} \frac{\partial}{\partial r}\log|f(re^{i\theta})|\,d\theta\,dr. \tag{2.12}$$

We estimate the integral on the right-hand side by first performing the r-integration. For fixed θ the set of r over which we have to integrate consists of disjoint intervals. If $\rho e^{i\theta} \in D_A$, then the value of $\log|f(re^{i\theta})|$ at the right-hand endpoint of one interval is $\log|f(\rho e^{i\theta})|$. If $ce^{i\theta} \in D_A$, then the value of $\log|f|$ at one left-hand endpoint is $\log|f(ce^{i\theta})|$. At all other endpoints $\log|f|$ has the value $\log A$. Hence

$$\int_{re^{i\theta} \in D(\rho)} \frac{\partial}{\partial r}\log|f(re^{i\theta})|\,d\theta \leqslant \log^+|f(\rho e^{i\theta})/A|$$

(where we set $|f(\rho e^{i\theta})| = A$ outside D_A).

Thus, using (2.12),

$$2\pi\int_c^\rho \frac{\nu(r)}{r}\,dr \leqslant \int_0^{2\pi}\log^+|f(\rho e^{i\theta})/A|\,d\theta + K\log(\rho/c). \tag{2.13}$$

If (2.1) holds, then we can find arbitrary large ρ such that (2.13) implies

$$2\pi\int_c^\rho \frac{\nu(r)}{r}\,dr < (p + K)\log\rho + K_1. \tag{2.14}$$

Since $\nu(r)$ is a nondecreasing function of r, (2.14) is only possible for arbitrarily large ρ if $\nu(r) \leqslant p + K$, i.e. the number of components of (2.3) is bounded by $p + K$. Since under our present assumptions each component is bounded, one can therefore find a ρ such that

$$\{\rho < |z| < \infty\} \subset D_A.$$

In $|z| > \rho$, $f(z)$ can be expanded in a Laurent series $f(z) = \Sigma_{-\infty}^{\infty} c_n z^n$,

$$|c_n| = \left|\frac{R^{-n}}{2\pi}\int_{|z|=R} f(Re^{i\theta})e^{-in\theta}\,d\theta\right| \qquad (R > \rho). \tag{2.15}$$

If (2.1) is true, (2.15) implies $|c_n| < R^{p-n}$ for arbitrarily large R and therefore $c_n = 0\ (n > p)$.

Therefore $f(z)$ either has a pole at ∞ and (b) of Theorem 1 holds or ∞ is a removable singularity of $f(z)$ and (a) holds by the maximum modulus principle.

3. Proof of Theorem 3. We shall use the following three properties of regular and irregular boundary points [2, Theorem III-34, p. 81; Theorem III-36, p. 82; Theorem III-38, p. 83]:

(1) The boundary point ζ of the region B is regular if and only if the Green's function $g(z, z_0)$ of B with pole at z_0 tends to 0 as $z \to \zeta$, $z \in B$.

(2) Regularity is a local property: If ζ is a boundary point of the regions B and C, both with boundaries of positive capacity, and if there is a neighborhood N of ζ such that $N \cap B = N \cap C$, then ζ is regular (or irregular) for both B and C.

(3) The set of irregular boundary points has capacity zero.

(i) (1.3) *does not hold, if the complement E of B in* $\mathbf{C}$ *has logarithmic capacity zero.*

PROOF. The set E has an Evans potential [2, p. 75]

$$v(z) = \int_E \log \frac{1}{|z - \zeta|}\, d\mu(\zeta), \qquad \mu(E) = 1, d\mu \geq 0 \tag{3.1}$$

such that, for every $\zeta \in E$, $v(z) \to \infty$ $(z \to \zeta)$.

The function $v(z)$ is superharmonic and, near ∞, $v(z) + \log|z|$ is harmonic.

Consider

$$u(z) = -\log|z| - v(z) + \log|z_0| + v(z_0) + 1,$$

where z_0 is some point in B. $u(z)$ is subharmonic in $|z| > 0$ and $u(z_0) = 1 > 0$.

For every boundary point $\zeta \neq 0$ of B, $\lim_{z\to\zeta} u(z) = -\infty$.

Near 0, $u(z) < \log|z|^{-1} + K$. Therefore one can find a constant A such that

$$u(z) < A \log|z|^{-1} \qquad \left(z \in B, |z| < \tfrac{1}{2}\right)$$

(maximum modulus principle applied to $u - A \log|z|^{-1}$ in $(\varepsilon < |z| < \frac{1}{2}) \cap B$). Hence $u(z)$ is a counterexample to the implication (1.3).

We can now assume that E has positive capacity and therefore the Green's function $g(z, z_0)$ of B with pole at z_0 exists.

Let $B_\rho = B \cup \{|z| < \rho\}$.

For sufficiently small ρ the complement of B_ρ has positive capacity, since otherwise

$$0 < \text{capacity } E = \text{capacity}(E \cap \{|z| \leq \rho\})$$
$$\leq \text{capacity}\{|z| \leq \rho\} = \rho.$$

Let $G(z, z_0)$ be the Green's function of B with pole at z_0. Since $B_\sigma \subset B_\rho$ $(0 < \sigma < \rho)$,

$$0 < G_\sigma(z, z_0) \leq G_\rho(z, z_0) \qquad (\sigma < \rho;\ z, z_0 \in B_\sigma).$$

Therefore $H(z, z_0) = \lim_{\rho\to 0} G(z, z_0)$ exists for $z \in B$, $z_0 \in B \cup \{0\}$. By Harnack's Theorem $H(z, 0)$ is a nonnegative harmonic function in B.

At every regular boundary point $\zeta \neq 0$ of B

$$0 \leq \limsup_{\substack{z\to\zeta \\ z\in B}} H(z, 0) \leq \limsup_{\substack{z\to\zeta \\ z\in B}} G_\rho(z, 0) = 0,$$

i.e.

$$\limsup_{\substack{z\to\zeta \\ z\in B}} H(z, 0) = 0. \tag{3.2}$$

By the definition of the Green's function

$$G_\rho(z, 0) < \log(1/|z|) + K(\rho) \qquad \left(|z| \leqslant \tfrac{1}{2}\rho,\ z \in B\right).$$

Therefore for all sufficiently small δ

$$H(z, 0) \leqslant G_\rho(z, 0) < 2\log|z|^{-1} \qquad (|z| = \delta,\ z \in B). \tag{3.3}$$

Also we can find a constant $C > 2$ such that

$$H(z, 0) \leqslant G_\rho(z, 0) < C\log|z|^{-1} \qquad \left(|z| = \tfrac{1}{2},\ z \in B\right). \tag{3.4}$$

By the extended maximum principle [2, Theorem III-28, p. 77] applied to $H(z, 0) + C\log|z|$ in $\{\delta < |z| < \frac{1}{2}\} \cap B$, it follows from (3.2)–(3.4) that

$$H(z, 0) \leqslant C\log|z|^{-1} \qquad \left(|z| < \tfrac{1}{2},\ z \in B\right). \tag{3.5}$$

(ii) (1.3) *holds if and only if* $H(z, 0) \equiv 0$ $(z \in B)$.

PROOF. The condition is necessary: If $H(z, 0)$ is not identically zero, then $H(z, 0) > 0$ in B. If every boundary point of B is regular, then $H(z, 0)$ is a counterexample to (1.3), by (3.2) and (3.5).

If the set F of irregular boundary points is not empty, then it has an Evans potential $v(z)$ given by (3.1) with E replaced by F. Let d be the diameter of ∂B. Then

$$\varphi(z) = \int_F \log\frac{d}{|z - \zeta|}\, d\mu(\zeta)$$

tends to ∞ as z tends to a point of F, $\varphi(z) > 0$ $(z \in \partial B)$ and $\psi(z) = \varphi(z) + g(z, \infty)$ is harmonic in B and

$$\liminf_{\substack{z\to\zeta\\ z\in B}} \psi(z) \geqslant 0 \qquad (\zeta \in \partial B).$$

The function $H(z, 0) - \varepsilon\psi(z)$ now provides a counterexample to (3), if ε is chosen so small that, for some $z_0 \in B$, $H(z_0, 0) - \varepsilon\psi(z_0) > 0$.

The condition is sufficient: Let $u(z)$ satisfy the hypotheses of Theorem 3, including that of (1.3). Then

$$v_\rho(z) = u(z) - 2CG_\rho(z, 0)$$

is subharmonic in B, and at all boundary points ζ of B other than 0

$$\limsup_{\substack{z\to\zeta\\ z\in B}} v_\rho(z) \leqslant \limsup_{\substack{z\in\zeta\\ z\in B}} u(z) \leqslant 0.$$

In $|z| \leqslant \frac{1}{2}\rho$, $G_\rho(z, 0) > \log|z|^{-1} - K_1(\rho)$. Hence

$$v_\rho(z) < 2CK_1(\rho) + C\log|z| \qquad \left(|z| \leqslant \tfrac{1}{2},\ z \in B\right)$$

and $\limsup_{z\to 0;\ z\in B} v_\rho(z) = -\infty$.

By the maximum principle $v_\rho(z) \leqslant 0$ $(z \in B)$. The conclusion $u(z) \leqslant 0$ $(z \in B)$ follows on letting ρ tend to 0, since

$$\lim G_\rho(z, 0) = H(z, 0) \equiv 0 \qquad (z \in B).$$

(iii) $H(z, 0) \equiv 0$ *in* B *if and only if* 0 *is a regular boundary point of* B.

PROOF. *Sufficiency*. We show first that

$$H(z, z_0) = g(z, z_0) \qquad (z, z_0 \in B). \tag{3.6}$$

Since $B_\rho \supset B$, $G_\rho(z, z_0) - g(z, z_0)$ is a nonnegative, bounded harmonic function in B. At every regular boundary point $\zeta \neq 0$ of B

$$\lim_{\substack{z \to \zeta \\ z \in B}} (G_\rho(z, z_0) - g(z, z_0)) = 0 \qquad (\rho < \rho(\zeta)).$$

Therefore

$$w(z) = H(z, z_0) - g(z, z_0) = \lim_{\rho \to 0} (G_\rho(z, z_0) - g(z, z_0))$$

is a nonnegative, bounded harmonic function in B and $\lim_{z \to \zeta;\ z \in B} w(z) = 0$ ($\zeta \in \partial B, \zeta \neq 0, z \in B$).

By the extended maximum principle $w(z) \leqslant 0$ ($z \in B$). But $w(z)$ is nonnegative, $w(z) = 0$ ($z \in B$); (3.6) is proved.

We choose $z_0 \in B$. Since the irregular boundary points of B form a set of capacity zero, one can find arbitrarily small δ such that $|z| = \delta$ does not contain any irregular points [**2**, p. 85].

The functions

$$f_\rho(z) = \begin{cases} G_\rho(z, z_0) & (z \in B), \\ 0 & (z \notin B) \end{cases}$$

are continuous on such a circle $|z| = \delta$. They converge to the continuous function $g(z, z_0)$ ($g(z, z_0) = 0$, $z \notin B$) on $|z| = \delta$. Since they are decreasing with ρ, they converge uniformly to $g(z, z_0)$ on $|z| = \delta$, as $\rho \to 0$. Therefore one can find $\rho = \rho(\delta, \varepsilon)$ such that

$$g(z, z_0) \leqslant G_\rho(z, z_0) \leqslant g(z, z_0) + \varepsilon \qquad (|z| = \delta, z \in B).$$

If $z = 0$ is a regular boundary point, we can choose δ so small that, given $\varepsilon > 0$,

$$g(z, z_0) < \varepsilon \qquad (|z| = \delta, z \in B),$$
$$G_\rho(z, z_0) < 2\varepsilon \qquad (|z| = \delta, z \in B)$$

for suitable $\delta = \delta(\varepsilon) < \frac{1}{2}|z_0|$, $\rho = \rho(\varepsilon)$.

The function $G_\rho(z, z_0)$ is harmonic and bounded above in $\{|z| < \delta\} \cap B_\rho$.

At all regular boundary points of B_ρ in $|z| < \delta$, $G_\rho(z, z_0)$ has the boundary value 0.

Therefore, by the extended maximum principle, $G_\rho(0, z_0) < 2\varepsilon$.

By the symmetry of the Green's function $G_\rho(z_0, 0) < 2\varepsilon$ for some $\rho > 0$. Hence

$$\lim_{\rho \to 0} G_\rho(z, z_0) = H(z_0, 0) = 0 \qquad (z_0 \in B).$$

Necessity. If $H(z, 0) \equiv 0$ ($z \in B$), then

$$G_\rho(0, z_0) = G_\rho(z_0, 0) < \varepsilon \qquad (\rho = \rho(\varepsilon)).$$

By continuity $G_\rho(z, z_0) < 2\varepsilon$ ($|z| < \eta(\varepsilon)$, $\rho = \rho(\varepsilon)$). Hence

$$g(z, z_0) \leqslant G_\rho(z, z_0) < 2\varepsilon \qquad (|z| < \eta(\varepsilon), z \in B).$$

That is to say $\lim g(z, z_0) = 0$ as $z \to 0$, $z \in B$. Therefore 0 is a regular boundary point.

This completes the proof of Theorem 3.

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Department of Mathematics, Cornell University, Ithaca, New York 14853